Lec 36: 11/20/2009

Symmetries and Their Consequences:

Our goal for a given system is to solve its equation(s) of motion and find its evolution over time. In general, solutions of the equations of motion have no specific reation to each other. However, it may happen that they are related to each other in a clear and meaningful way. Then, starting with one such solution, one can generate a whole class of solutions via some transformations. This happens if the Hamiltonian of the system is invariant under some transformations. These transformations, acting on a given solution, will then generate a class of solutions with the same energy. These transformations form a group, called symmetry group. The elements of this group are operators. In quantum mechanics, these operators operate on the state vectors of a system that belong to a Hilbert space.

There are two types of symmetries:

1- Spacetime symmetries: Under these symmetries spacetime coordinates are transformed. Examples are translation, rotation, parity, etc.

2- Internal symmetries: Under these symmetries only the wavefunctions are transformed. Internal symmetries arise because of some internal quantum numbers that have nothing to do with spacetime transformation. An example is the electric charge.

Our focus here is on spacetime symmetries. These symmetries are divided in two groups,

(a) Continuous symmetries: In this case any transformation can be obtained by a continuous change in some parameters. As an example, consider translation in one spatial dimension. This is a one parameter group;

$x \to x + a$ Translation by amount a

"a" is arbitrary. Any translation can be obtained by continuously changing the parameter "a" ($a=0$ corresponds to identity).

(b) Discrete symmetries; In this case transformations (or some of them) cannot be obtained by a continuous change in some parameter. For example, consider parity in one dimension;

$x \to -x$

Since the sign flips, we cannot get parity transformation by continuously varying a parameter.

Both of the continuous (translation, time translations, rotation) and discrete (parity, time reversal) spacetime symmetries are important. Here we focus on the continuous spacetime symmetries.

Translation in One Dimension:

The operation of translation on the position eigenket is defined as,

$$T_{(a)}\,|x\rangle = |x+a\rangle$$

It turns out that $T_{(a)}\,T_{(-a)} = T_{(-a)}\,T_{(a)} = I$, thus,

$$T^{-1}_{(a)} = T_{(-a)}$$

Also,

$|x\rangle$ and $|x+a\rangle$ are normalized

$$\langle x+a|x+a\rangle = \langle x|\,T^{\dagger}_{(a)}\,T_{(a)}\,|x\rangle \Rightarrow T^{\dagger}_{(a)}\,T_{(a)} = 1$$

$$\Rightarrow T^{\dagger}_{(a)} = T^{-1}_{(a)}$$

Therefore translation is a unitary operator.

Now, since we know how T changes position eigenstates (that form an orthonormal basis), we can also find how it transforms an arbitrary state vector $|\psi\rangle$.

Note that:

$\psi(x) = \langle x|\psi\rangle$

$x \rightarrow x+a \Rightarrow \psi(x) \rightarrow \psi(x-a)$

In other words, translating x by amount "a" is equivalent to shifting the argument of function $\psi(x)$ by amount "$-a$"; the point $x=0$ after translation is actually the point $x=-a$.

Writing dow the series expansion for $\psi(x-a)$:

$$\psi(x-a) = \langle x-a|\psi\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}(-a)^n \frac{d^n\psi}{dx^n}(x) =$$

$$\langle x|\frac{1}{n!}(-a)^n\left(\frac{i}{\hbar}\right)^n P^n|\psi\rangle = \langle x|\exp\left(\frac{-i}{\hbar}Pa\right)|\psi\rangle$$

P → momentum operator

On the other hand:

$\langle x-a|\psi\rangle = \langle x|T(a)|\psi\rangle$

Hence:

$T(a) = \exp\left(-\frac{i}{\hbar}Pa\right)$

Momentum operator P is the generator of translation. This is understanble intuitively because translation implies motion of the particle, and hence momentum. (particle with zero momentum will stay at the same place).

Translational invariance means that if $|\psi\rangle$ is a solution of the Schrodinger equation, so will $T_{(a)}|\psi\rangle$ be:

$$H|\psi_{(t)}\rangle = +i\hbar \frac{\partial}{\partial t}|\psi_{(t)}\rangle \Rightarrow \underset{\forall a \in R}{H\, T_{(a)}|\psi_{(t)}\rangle = +i\hbar \frac{\partial}{\partial t} T_{(a)}|\psi_{(t)}\rangle}$$

Multiplying both sides by T^{-1}, we find that:

$$T^{\dagger}_{(a)} H\, T_{(a)} |\psi_{(t)}\rangle = i\hbar \frac{\partial}{\partial t}|\psi_{(t)}\rangle$$

Note that $T^{-1} = T^{\dagger}$, and $T_{(a)}$ has no time dependence.

The equality holds at all times only if:

$$T^{\dagger}_{(a)} H\, T_{(a)} = H \qquad \forall a \in R$$

$$\Rightarrow \boxed{[H, P] = 0}$$

Therefore, translational invariance is equivalent to having $[H, P]=0$. For a one particle system this essentially means a constant potential.

As advertized, we find that translational invariance (translation is a continuous transformation) gives rise to a conserved quantity. Recall from Ehrenfest's theorem that:

$$\boxed{\langle \dot{P} \rangle = -\frac{i}{\hbar}\langle [H, P] \rangle = 0 \Rightarrow \langle P \rangle = \text{Const.}}$$

Translation in Three dimensions:

Translation in three dimensions has three parameters.

$$(x, y, z) \longrightarrow (x+a, y+b, z+c)$$

$$\psi(x, y, z) \rightarrow \psi(x-a, y-b, z-c)$$

The translation operator $T(a, b, c)$ is:

$$T(a, b, c) = \exp\left[-\frac{i}{\hbar}(P_x a + P_y b + P_z c)\right]$$

Since $[P_x, P_y] = [P_y, P_z] = [P_x, P_z] = 0$, we have:

$$T_{(a,b,c)} = \exp\left(-\frac{i}{\hbar} P_x a\right) \exp\left(-\frac{i}{\hbar} P_y b\right) \exp\left(-\frac{i}{\hbar} P_z c\right)$$

And we have three conserved quantities for a translationally invariant system in three dimensions,

$$\boxed{[H, P_x] = [H, P_y] = [H, P_z] = 0}$$

$$\Rightarrow \langle \dot{P}_x \rangle = \langle \dot{P}_y \rangle = \langle \dot{P}_z \rangle = 0$$

$\boxed{\Rightarrow \langle P_x \rangle, \langle P_y \rangle, \langle P_z \rangle \text{ are constant (not the same)}}$